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LETTER TO THE EDITOR

Flux infiltration into soils: analytic solutions

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Abstract. The Burgers model for flux infiltration into a semi-infinite soil is solved for generic space-dependent initial moisture and time dependent supply rate. General formulae in terms of quadratures are obtained for the moisture content and for the time to ponding of the soil.

The vertical non-hysteretic flow of water into soils has been described by Burgers equation in the case both of rigid soils [1] and of swelling soils [2].

Indeed, the Burgers model provides analytic solutions which are in good agreement with the data obtained by experimental and numerical simulations, relative to the case of rigid (non-swelling) soils and to the preponding regime of moderately swelling soils [1, 2].

Such agreement justifies the approximations of a constant diffusivity and a quadratic conductivity–water constant relationship, which characterize the Burgers model.

However, all previous analysis treat only the case of a constant water supply rate at the surface, and of a constant (and special) value for the initial moisture content.

In this letter we extend the formalism to the case of generic time-dependent rate and space-dependent initial moisture. We obtain explicit formulae (involving only quadratures) for the moisture content and for the time to ponding of the soil in this general case. As a particular example, the case of a linearly increasing rate is compared with the (known) case of a constant rate, in the situation characterized by an initially constant moisture (in the vertical direction), having the value at which the hydraulic conduction vanishes (see below). We start with the flow equation describing the non-hysteric water infiltration in a rigid or slowly swelling semi-infinite one-dimensional solid medium [1, 2]:

$$\vartheta_t = D\vartheta_{zz} - \frac{\mathrm{d}K}{\mathrm{d}\vartheta}\vartheta_Z \qquad \vartheta = \vartheta(z, t) \tag{1}$$

where t is time and z is depth. In (1) ϑ is the dimensionless moisture ratio

$$\vartheta = \varphi/(1-\varphi) \tag{2}$$

with φ denoting the volumetric water constant.

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The diffusivity D is assumed to be constant, and the hydraulic conductivity $K(\vartheta)$ to obey the quadratic law

$$K(\vartheta) = -\tfrac{1}{2}A(\vartheta - \bar{\vartheta})^2 \tag{3a}$$

$$\frac{\mathrm{d}K}{\mathrm{d}\vartheta} = -A(\vartheta - \bar{\vartheta}) \tag{3b}$$

with A and $\bar{\vartheta}$ positive constants. The value $\bar{\vartheta}$ corresponds to the position of the maximum in the $K(\vartheta)$ curve [2] and depends on the particular soil under consideration. At the free surface of the soil, $z = 0$, we assume that the water supply flux is a given function of time

$$[-D\vartheta_2 + K(\vartheta)]|_{z=0} = R(t) \qquad t \geqslant 0. \tag{4}$$

Moreover, we supplement (1) with a space dependent initial condition

$$\vartheta(z, 0) = \vartheta_0(z) \qquad z \geqslant 0. \tag{5}$$

Here $\vartheta_0(Z)$ is the antecedent water moisture; its value could be smaller or greater than $\bar{\vartheta}$, depending on the particular situation antecedent to the onset of the water supply. Insertion of (3*b*) into (1) yields the evolution equation

$$\vartheta_t = D\vartheta_{zz} + A(\vartheta - \bar{\vartheta})\vartheta_z \tag{6}$$

with boundary conditions (from (3*a*) and (4))

$$D\vartheta_z + \tfrac{1}{2}A(\vartheta - \bar{\vartheta})^2 = -R(t) \qquad \text{at } z = 0. \tag{7}$$

We now introduce the dimensionless variables

$$\zeta = \frac{A\delta}{2D}z \qquad \tau = \frac{A^2\delta^2}{4D}t \tag{8a}$$

with

$$\delta = \vartheta_s - \bar{\vartheta} \tag{8b}$$

where ϑ_s is a notional upper moisture value, corresponding to saturation (see below); moreover, we set

$$\Psi(\zeta, \tau) = [\vartheta(z, t) - \bar{\vartheta}]/\delta. \tag{8c}$$

In terms of the dimensionless variables (8), the evolution equation (6) reads

$$\Psi_\tau = \Psi_{\zeta\zeta} + 2\Psi\Psi_\zeta \tag{9}$$

with boundary condition

$$\Psi^2(0, \tau) + \Psi_\zeta(0, \tau) = -F(\tau) \qquad \tau \geqslant 0 \tag{10a}$$

$$F(\tau) = [2/(A\delta^2)]R(t) \tag{10b}$$

and initial condition

$$\Psi(\zeta, 0) = \Psi_0(\zeta) = [\vartheta_0(z) - \bar{\vartheta}]/\delta \qquad \zeta \geqslant 0. \tag{11}$$

The initial/boundary value problem (9–11) is now solved by introducing the generalized Hopf–Cole transformation [3]

$$\varphi(\zeta, \tau) = C(\tau)\Psi(\zeta, \tau)\exp\left[\int_0^\zeta d\zeta' \, \Psi(\zeta', \tau)\right] \tag{12a}$$

$$\Psi(\zeta, \tau) = \varphi(\zeta, \tau) \bigg/ \left(C(\tau) + \int_0^\zeta d\zeta' \, \varphi(\zeta', \tau)\right) \tag{12b}$$

$$C(0) = 1. \tag{12c}$$

It is easily seen that this transformation maps (9) into

$$\varphi_\tau = \varphi_{\zeta\zeta} \tag{13a}$$

$$\dot{C}(\tau) = \varphi_\zeta(0, \tau). \tag{13b}$$

Moreover, the boundary condition (10*a*) and the initial condition (11) are mapped into the boundary condition

$$\varphi_\zeta(0, \tau) = C(\tau)(\Psi_\zeta(0, \tau) + \Psi^2(0, \tau)) = -C(\tau)F(\tau) \tag{14}$$

and into the initial condition

$$\varphi(\zeta, o) = \varphi_0(\zeta)\exp\left[\int_0^\zeta d\zeta' \, \Psi_0(\zeta')\right]. \tag{15}$$

It is immediately seen that (13*b*) and (14) imply for the unknown function $C(\tau)$ the evolution equation

$$\frac{\dot{C}(\tau)}{C(t)} = -F(\tau) \tag{16}$$

which, once integrated with the initial condition (12*c*), gives

$$C(\tau) = \exp\left[-\int_0^\tau d\tau' \, F(\tau')\right]. \tag{17}$$

The solution $\varphi(\zeta, \tau)$ of the linear equation (13*a*) with boundary condition (14) and initial condition (15), can explicitly be obtained via the Fourier transform (originally invented just for this purpose!). It reads [4]

$$\varphi(\zeta, \tau) = w(\zeta, \tau) - \int_0^\tau d\tau' \, [\pi(\tau - \tau')]^{-1/2}\dot{C}(\tau')\exp[-\zeta^2/4(\tau - \tau')] \tag{18}$$

with

$$w(\zeta, \tau) = (\pi\tau)^{1/2}\int_0^\infty d\eta \, (\zeta\eta/2\tau)\exp[-(\zeta^2 + \eta^2)/4\tau]\varphi_0(\eta) \tag{19}$$

with $C(\tau)$ give by (17) and $\varphi_0(\eta)$ given by (15).

The solution $\varphi(\zeta, \tau)$ of the nonlinear equation (9) can finally be recovered from (12*b*), (17), (18) and (19).

We now turn our attention to the determination of the time to ponding t_p. The phenomenon of ponding at the soil surface can be observed for example during rainfall, when the rainfall rate exceeds the transport rate of water away from the soil surface; the latter is governed by the nonlinear convection term in (1).

In our formalism, after some time $t = t_p$ from the onset of the water supply the moisture ratio at the soil surface $z = \zeta = 0$ will eventually attain the saturation level ϑ_s; ponding of free water will then occur at the soil surface for $t > t_p$.

In our dimensionless units, we obtain from (8*c*) and (8*b*)

$$\Psi(0, \tau_p) = \frac{\vartheta_s - \bar{\vartheta}}{\delta} = 1. \tag{20}$$

Equation (12*a*) then gives

$$\varphi(0, \tau_p) = C(\tau_p) \tag{21}$$

which in turn implies via (18) and (19)

$$w(0, \tau_p) = \int_0^{\tau_p} \mathrm{d}\tau\, [\pi(\tau_p - \tau)]^{-1/2} \dot{C}(\tau) \tag{22a}$$

with

$$w(0, \tau_p) = (\pi\tau_p)^{-1/2} \int_0^{\infty} \mathrm{d}\eta\, \exp(-\eta^2/4\tau_p)\varphi_0(\eta). \tag{22b}$$

From these formulae there obtains the equation that determines the time to ponding τ_p:

$$\begin{aligned}(\pi\tau_p)^{-1/2} &\int_0^{\infty} \mathrm{d}\eta\, \exp(-\eta^2/4\tau_p)\varphi_0(\eta) \\ &= \exp\left[-\int_0^{\tau_p} \mathrm{d}\tau F(\tau)\right] - \int_0^{\tau_p} \mathrm{d}\tau\, [\pi(\tau_p - \tau)]^{-1/2} F(\tau) \exp\left[-\int_0^{\tau} \mathrm{d}\tau' F(\tau')\right].\end{aligned} \tag{23}$$

Let us note that this equation contains as input data only the initial moisture configuration $\varphi_0(\zeta)$ (see (15), (11) and (8)) and the rate of water supply at the surface $F(\tau)$ (see (10) and (8)).

In this special case in which the initial condition for the moisture of the soil is constant and coincides with the value $\bar{\vartheta}$ at which the hydraulic conductivity $K(\vartheta)$ vanishes (see (3)), so that

$$\vartheta_0(\zeta) = 0 \tag{24}$$

(see (15) and (11)), the formula (23) can be rewritten in the following neater form:

$$\int_0^{\tau_p} \mathrm{d}\tau [\pi(\tau_p - \tau)]^{-1/2} F(\tau) \exp\left[\int_{\tau}^{\tau_p} \mathrm{d}\tau' F(\tau')\right] = 1. \tag{25}$$

This is the only case treated previously, with the additional assumption that the water supply flux $R(\tau)$ be time-independent

$$R(\tau) = \bar{R}. \tag{26a}$$

In this case (see 10b)

$$F(\tau) = \bar{F} = [2/(A\delta^2)]\bar{R} \tag{26b}$$

is also constant, and (25) reduces, of course, to the previously formula [2]

$$-i(\bar{F})^{1/2}\mathrm{Erf}[i(\bar{F}\tau_{\mathrm{p}})^{1/2}] = 1 \tag{27a}$$

where

$$-i\mathrm{Erf}(iy) = 2\pi^{-1/2}\sum_{n=0}^{\infty} y^{2n+1}[(2n+1)n!]^{-1} \tag{27b}$$

is a regular, real function of y.

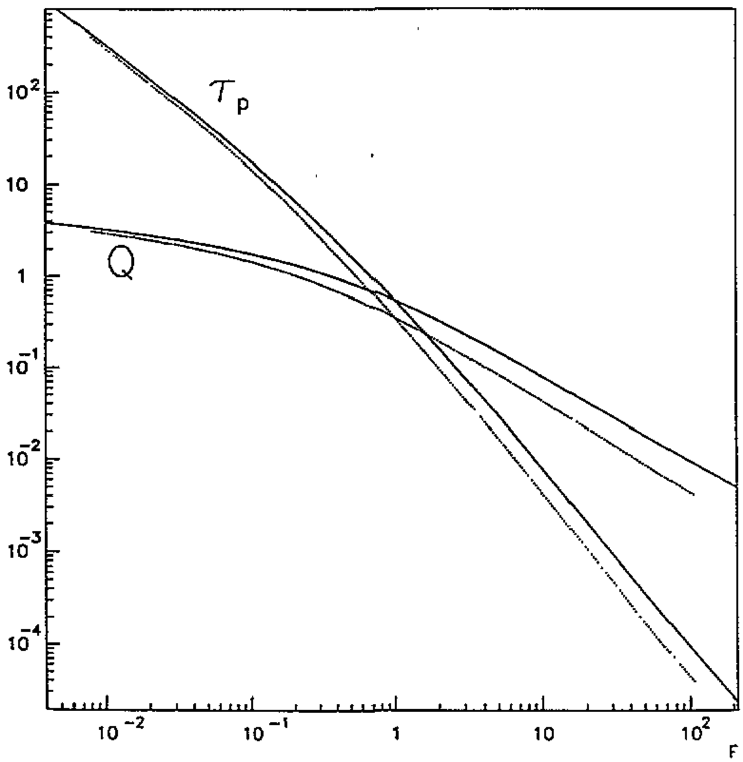

Figure 1. The time to ponding τ_{p} and the total amount of fallen water Q as a function of the average water flux $\bar{F}$. Dotted line: constant flux (see (26)). Solid line: linearly increasing flux (see (28)).

To make a comparison with the previously known case, we keep the condition (24) but we assume the water supply rate to grow linearly from an initially vanishing value:

$$R(\tau) = k\tau \tag{28a}$$

$$F(\tau) = k_1\tau \tag{28b}$$

$$k_1 = [2/(A\delta^2)]k. \tag{28c}$$

Then (24) yields

$$k_1 \int_0^{\tau_p} \mathrm{d}\tau\, [\pi(\tau_p - \tau)]^{-1/2} \tau \exp[\tfrac{1}{2}k_1(\tau_p^2 - \tau^2)] = 1. \tag{29}$$

Equations (27) and (29) can be easily solved numerically. To compare the two cases, we plot in figure 1 the time to ponding τ_p as a function of the average water flux $\bar{F}$ up to ponding

$$\bar{F} = \tau_p^{-1} \int_0^{\tau_p} \mathrm{d}\tau\, F(\tau). \tag{30}$$

Of course in the case of a constant flux, $\bar{F}$ coincides with the value (26*b*).

In figure 1 we also plot for the two cases the total amount Q of the water fallen up to the time τ_p,

$$Q = \bar{F}\tau_p.$$

As expected, figure 1 shows that the time to ponding τ_p is a rapidly decreasing function of the water amount at the inlet. The decrease, however, is faster in the case of a constant supply rate.

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